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One-dimensional percolation problem with further neighbour bonds—transfer-matrix approach

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Abstract. The transfer-matrix method is used to find the exact expressions for the correlation length in the critical region for both one-dimensional site and bond percolation problems with bonds connecting Lth-nearest neighbours (for any finite L). For the site percolation, the correlation length exponent ν is found to be L, consistent with the result obtained from the generating function method. For the bond percolation, we find $\nu = L(L+1)/2$.

1. Introduction

In the percolation problem, the one-dimensional system is one of the few cases where exact solutions can be found (for reviews see Stauffer 1979 and Essam 1980). One-dimensional site percolation with bonds connecting Lth-nearest neighbour sites has been solved by the generating function method (Klein *et al* 1978). Although the critical occupation probability p_c is trivial in one dimension (p_c is always equal to 1), the critical exponents are found to be L dependent. In particular, the correlation length exponent ν is found to be equal to L. Such a 'bond range' dependence of the critical behaviour is closely related to the corresponding 'thermal' problem with multi-spin interactions (Klein *et al* 1978).

Recently, the transfer-matrix method has been used to find the exact critical behaviour for both one-dimensional site and bond percolation systems with further neighbour bonds (Zhang and Shen 1982). In that work, the authors showed that the transfer-matrix method gives the consistent result $\nu = L$ in the site percolation case. For the bond percolation, much richer critical behaviour was found. If all the Lth-nearest neighbour bonds have equal occupation probability, ν is found to be $L(L+1)/2$. However, in that work, the calculations are only done for small L (L up to 3); no proof has been given for the case of general L. In this work, we give a systematic extension of the transfer-matrix method to one-dimensional percolation systems from previous low L values to any finite L.

2. Transfer matrix

Consider a linear chain with bonds connecting Lth-nearest neighbours (cf figure 1). Following the method used by Zhang and Shen (1982), we divide the chain into overlapping columns each containing L sites. If we take the sites $1, 2, \ldots, L-1$ and

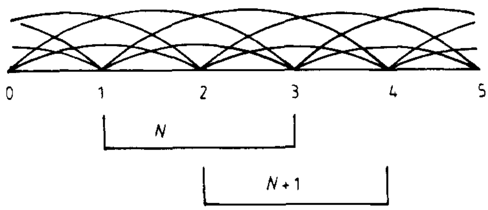

Figure 1. A linear chain with bonds connecting 3rd-nearest neighbours ($L = 3$). Sites (1, 2, 3) and (2, 3, 4) are taken to be the Nth and $(N+1)$th columns respectively.

L as the Nth column, then the $(N+1)$th column contains sites $2, 3, \ldots, L$ and $L+1$. Since each site can be either connected or disconnected to the first column, there are 2^L configurations in each column. The transfer matrix transfers the probability distribution of various configurations in the Nth column to the $(N+1)$th column. The correlation length ξ is related to the largest non-trivial eigenvalue λ_m of the transfer matrix by the relation (Derrida and Vannimenus 1980)

$$\xi = -1/\ln \lambda_m. \tag{1}$$

From (1), the critical point and the correlation length exponent ν can be found (Zhang and Shen 1982).

To each ith site of the Nth column we assign a value m_i which has the value 1 or 0 depending on whether the ith site is connected or disconnected to the first column. So, any configuration in the Nth column can be represented by $(m_1, m_2, \ldots, m_L)$. If $P_{m_1 m_2 \ldots m_L}(N)$ is the probability of being in the configuration $(m_1, m_2, \ldots, m_L)$, then the transfer matrix is defined by

$$P_{n_1 n_2 \ldots n_L}(N+1) = \sum_{m_1, \ldots, m_L = 0,1} \langle n_1, \ldots, n_L | T | m_1, \ldots, m_L \rangle P_{m_1 m_2 \ldots m_L}(N), \tag{2}$$

where similar notation $(n_1, \ldots, n_L)$ is used in the $(N+1)$th column. For the site percolation, we have

$$\begin{aligned} &\langle n_1, \ldots, n_L | T | m_1, \ldots, m_L \rangle \\ &\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L} (q\delta_{n_L 0} + p\delta_{n_L 1}) \qquad \text{if } (m_1, \ldots, m_L) \neq (0, \ldots, 0), \\ &\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L}, \delta_{n_L 0} \qquad \text{if } (m_1, \ldots, m_L) = (0, \ldots, 0), \end{aligned} \tag{3}$$

where p is the site occupation probability and $q = 1 - p$, and δ is the Kronecker delta.

For the bond percolation, we have

$$\begin{aligned} &\langle n_1, \ldots, n_L | T | m_1, \ldots, m_L \rangle \\ &\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L} [q_L^{m_1} \ldots q_1^{m_L} \delta_{n_L 0} \\ &\qquad + (1 - q_L^{m_1} \ldots q_1^{m_L}) \delta_{n_L 1}] \qquad \text{if } (m_1, \ldots, m_L) \neq (0, \ldots, 0), \\ &\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L}, \delta_{n_L 0} \qquad \text{if } (m_1, \ldots, m_L) = (0, \ldots, 0), \end{aligned} \tag{4}$$

where $q_i = 1 - p_i$ and p_i is the occupation probability of the ith-nearest neighbour bond.

Following Domb (1949), we label each configuration $(m_1, \ldots, m_L)$ and $(n_1, \ldots, n_L)$ of the Nth and $(N+1)$th columns by numbers m and n respectively, where m and n

are defined by

$$m = m_1 2^{L-1} + m_2 2^{L-2} + \ldots + m_{L-1} 2 + m_L, \tag{5}$$

$$n = n_1 2^{L-1} + n_2 2^{L-2} + \ldots + n_{L-1} 2 + n_L. \tag{6}$$

Since both (3) and (4) contain the factor $\delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L}$, the non-vanishing matrix elements of (3) and (4) can be found from the relation

$$n = 2(m - m_1 2^{L-1}) + n_L \tag{7}$$

where (5) and (6) have been used to obtain (7). Using the above labelling procedure, the transfer matrix $\langle n_1, \ldots, n_L | T | m_1, \ldots, m_L \rangle$ can be denoted by $T^{(L)}_{nm}$ and has a systematic duo-diagonal form (Domb 1949).

For site percolation, using (3) and (7), we find

$$T^{(L)}_{nm} = \begin{array}{l} (m) = \; 0 \;\; 1 \;\; 2 \quad \ldots \quad 2^{L-1}-1 \;\; 2^{L-1} \quad \ldots \quad 2^L - 1 \\ \left(\begin{array}{l}
1 \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \\
0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \\
0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \\
0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \\
0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \\
0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \\
0 \; 0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; q \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \\
0 \; 0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; p \; 0 \; 0 \; \ldots \; 0 \; 0 \; 0 \\
\qquad\qquad\qquad \ldots \\
\qquad\qquad\qquad \ldots \\
0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \; 0 \; 0 \\
0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p \; 0 \; 0 \\
0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \; 0 \\
0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p \; 0 \\
0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; q \\
0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; \ldots \; 0 \; 0 \; p
\end{array} \right) \end{array} . \tag{8}$$

If we define the function $A^{(L)}(\lambda, q)$ as

$$\det(T^{(L)}_{nm} - \lambda \delta_{nm}) \equiv (1 - \lambda) A^{(L)}(\lambda, q), \tag{9}$$

then the largest non-trivial eigenvalue of $T^{(L)}_{nm}$ is given by the largest root $\lambda_{\mathrm{m}}(q)$ of the equation $A^{(L)}(\lambda, q) = 0$. In the critical region, we fortunately do not have to find the exact expression of $\lambda_{\mathrm{m}}(q)$. Since the critical point can be located from the condition $A^{(L)}(\lambda = 1, q_c) = 0$, where the correlation length ξ diverges, it will be shown in the appendix that if $A^{(L)}(1, q)$ has the form bq^K, where b is a constant and K is a positive integer, then the correlation length ξ, in the critical region, is given by

$$\xi(q) = x b^{-1} q^{-K} \qquad \text{for small } q \tag{10}$$

where

$$x = \left. \frac{\partial A^{(L)}(\lambda, q)}{\partial \lambda} \right|_{\lambda = 1, q = 0} . \tag{11}$$

From (8) and (9), we can eliminate all the p's and q's in the right half of the determinant $A^{(L)}(1, q)$ by some manipulations. It can be verified that the following relation holds:

$$A^{(L)}(1, q) = q A^{(L-1)}(1, q). \tag{12}$$

Since $A^{(1)}(1,q)=-q$, we find $A^{(L)}(1,q)=-q^L$ and $q_c=0$ ($p_c=1$). The value of x of (11) can also be evaluated by a similar procedure and is found to be -1 for any finite L. From (10), we find, for small q, that $\xi=q^{-L}$ and $\nu=L$ (see Klein *et al* 1978). These are the exact results for any finite L.

In the case of bond percolation, $T_{nm}^{(L)}$ is also a duo-diagonal matrix of order 2^L. It is rather difficult to evaluate the determinant $A^{(L)}(1,q_1,\ldots,q_L)$ in the general case. However, when all the Lth-nearest neighbour bonds have the same occupation probability, the exact critical behaviour can also be obtained by using a symmetric representation as will be shown in the next section.

3. Symmetric representation

Both for the site percolation and the bond percolation with all Lth-nearest neighbour bonds having equal occupation probability, a symmetric representation can be used to find the exact critical behaviour. In this representation, we assume that all the configurations having the same number of sites r connecting to the first column have the same probability. We define $P_{\{m\},r}$, in the Nth column, as

$$P_{\{m\},r}(N)\equiv P_{m_1\ldots m_L}(N) \qquad \text{for all } \sum_{i=1}^{L} m_i=r \tag{13}$$

where r has the values $0,1,\ldots,L$. If we denote P_r as the total probability of all the configurations having r sites connecting to the first column, we have

$$P_r(N)\equiv \sum_{\substack{m_1,\ldots,m_L=0,1\\ m_1+\ldots+m_L=r}} P_{\{m\},r}=\binom{L}{r}P_{\{m\},r}. \tag{14}$$

Using the same definition of $P_s(N+1)$ for the $(N+1)$th column, from (2), we find

$$\begin{aligned}P_s(N+1)&=\sum_{r=0}^{L}\sum_{\substack{n_1,\ldots,n_L=0,1\\ n_1+\ldots+n_L=s\\ m_1,\ldots,m_L=0,1\\ m_1+\ldots+m_L=r}}\langle n_1,\ldots,n_L|T|m_1,\ldots,m_L\rangle\binom{L}{r}P_r(N)\\ &\equiv\sum_{r=0}^{L}M_{sr}^{(L)}P_r(N)\end{aligned} \tag{15}$$

where $M_{sr}^{(L)}$ is thus the reduced transfer matrix in the symmetric representation having dimensionality $(L+1)\times(L+1)$.

For the case of site percolation, substituting (3) into (15), after some manipulations, we find

$$\begin{aligned}M_{sr}^{(L)}&=\delta_{sr} \qquad \text{if } r=0,\\ &=\delta_{s+1,r}rq/L+\delta_{sr}[rp/L+(L-r)q/L]\\ &\quad+\delta_{s-1,r}(L-r)p/L \qquad \text{if } r=1,2,\ldots,L.\end{aligned} \tag{16}$$

Again, we define $A_{\text{sym}}^{(L)}(\lambda,q)$ as

$$\det(M_{sr}^{(L)}-\lambda\delta_{sr})\equiv(1-\lambda)A_{\text{sym}}^{(L)}(\lambda,q). \tag{17}$$

From (16) and (17), $A^{(L)}_{\rm sym}(1, q)$ can be evaluated exactly and is found to be

$$A^{(L)}_{\rm sym}(1, q) = F(L)q^L \tag{18}$$

with

$$F(L) = (-1)^L (L-1)!/L^{(L-1)}. \tag{19}$$

It can also be shown that the value of x of (11) is exactly equal to $F(L)$ of (19). Using (10) and (11), we obtain the same results as in the last section; namely, for small q, $\xi = q^{-L}$ and $\nu = L$.

In the case of bond percolation, if we assume that all the Lth-nearest neighbour bonds have equal occupation probability, from (4) and (15), with some manipulations, we find

$$\begin{aligned} M^{(L)}_{sr} &= \delta_{sr} \qquad \text{if } r = 0, \\ &= \delta_{s+1,r}\frac{rq^r}{L} + \delta_{sr}\left(\frac{r}{L}(1-q^r) + \frac{(L-r)}{L}q^r\right) \\ &\quad + \delta_{s-1,r}\frac{(L-r)}{L}(1-q^r) \qquad \text{if } r = 1, 2, \ldots, L. \end{aligned} \tag{20}$$

Substituting (20) into (17), $A^{(L)}_{\rm sym}(1, q)$ can again be evaluated exactly and is found to have the form

$$A^{(L)}_{\rm sym}(1, q) = F(L)q^{L(L+1)/2}, \tag{21}$$

where $F(L)$ is given by (19). In this case the value of x of (11) is again found to be $F(L)$ of (19). From (10) and (11), we obtain, for small q, that $\xi = q^{-L(L+1)/2}$ and $\nu = L(L+1)/2$.

In summary, we have used the transfer-matrix method to find the exact expressions for the correlation length in the critical region for both one-dimensional site and bond percolation systems with bonds connecting Lth-nearest neighbours. In the case of site percolation, the correlation length exponent ν is found to be L, consistent with the results obtained by using the generating function method. In the case of bond percolation with all the Lth-nearest neighbour bonds having equal occupation probability, ν is found to be $L(L+1)/2$. This confirms the previous prediction by Zhang and Shen (1982).

Recently, we have used the infinitely large cell to cell renormalisation group method proposed by Reynolds *et al* (1980) to treat the bond percolation case (Li *et al*). It is also found that $\nu = L(L+1)/2$, consistent with the results obtained here by using the transfer-matrix method.

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Appendix

For any transfer matrix $T_{nm}(q)$ of order $(J+1)$, the largest non-trivial eigenvalue is given by the largest root of the equation $A(\lambda, q) = 0$, where $A(\lambda, q)$, a determinant of order J, is defined by

$$\det(T_{nm} - \lambda\delta_{nm}) \equiv (1-\lambda)A(\lambda, q). \tag{A1}$$

In this appendix, we will show that if $A(1, q)$ has the form bq^K, where b is a constant and K is a positive integer, then the correlation length in the critical region (small q) has the form

$$\xi(q) = xb^{-1}q^{-K}, \tag{A2}$$

where

$$x = \left.\frac{\partial A(\lambda, q)}{\partial\lambda}\right|_{\lambda=1, q=0}. \tag{A3}$$

Since $A(\lambda, q)$ is a polynomial of λ of order J, we can write

$$A(\lambda, q) = \sum_{n=0}^{J} a_n(q)\lambda^n. \tag{A4}$$

If $\lambda_{\mathrm{m}}(q)$ is the largest root of the equation $A(\lambda, q) = 0$, then $\lambda_{\mathrm{m}}(q)$ must satisfy $\lambda_{\mathrm{m}}(0) = 1$ and

$$A(\lambda_{\mathrm{m}}(q), q) = \sum_{n=0}^{J} a_n(q)\lambda_{\mathrm{m}}^n(q) \equiv 0. \tag{A5}$$

Expanding $\lambda_{\mathrm{m}}(q)$ in the vicinity of the critical point $q = 0$, we have

$$\lambda_{\mathrm{m}}(q) = 1 + \lambda'_{\mathrm{m}}(0)q + \tfrac{1}{2}\lambda''_{\mathrm{m}}(0)q^2 + \ldots + \frac{1}{n!}\lambda_{\mathrm{m}}^{(n)}(0)q^n + \ldots. \tag{A6}$$

Taking the total derivatives on both sides of (A5), we have

$$\mathrm{d}^n A/\mathrm{d}^n q \equiv 0 \qquad \text{for all integer } n. \tag{A7}$$

When $n = 1$, we find

$$\sum_{n=0}^{J} [a'_n(q)\lambda_{\mathrm{m}}^n(q) + na_n(q)\lambda_{\mathrm{m}}^{n-1}(q)\lambda'_{\mathrm{m}}(q)] \equiv 0. \tag{A8}$$

Putting $q = 0$, $\lambda_{\mathrm{m}}(0) = 1$, and using the relation $\Sigma_{n=0}^{J} a_n(q) = A(1, q) = bq^K$, (A8) gives

$$\begin{aligned} \lambda'_{\mathrm{m}}(0) &= -x^{-1}bK && \text{if } K = 1, \\ &= 0 && \text{if } K > 1, \end{aligned} \tag{A9}$$

where

$$x = \sum_{n=1}^{J} na_n(0) \equiv \left.\frac{\partial A(\lambda, q)}{\partial\lambda}\right|_{\lambda=1, q=0}. \tag{A10}$$

If $K > 1$, we can take higher derivatives on both sides of (A8) and put $q = 0$, $\lambda_{\mathrm{m}}(0) = 1$ afterwards. In general, we find the following results:

$$\begin{aligned} \lambda_{\mathrm{m}}^{(n)}(0) &= 0 && \text{if } n < K, \\ &= -x^{-1}bK! && \text{if } n = K. \end{aligned} \tag{A11}$$

From (A6) and (A11), we obtain

$$\lambda_m(q) = 1 - x^{-1}bq^K + O(q^{K+1}). \tag{A12}$$

Substituting (A12) into (1), we finally arrive at (A2) and (A3).

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